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FINITE ELASTIC DEFORMATIONS OF THIN CYLINDRICAL TUBES BY MANDRELS WITH DISCONTINUOUS CURVATURE

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Abstract—A thin cylindrical elastic tube, reinforced on the outer surface by a two parameter family of inextensible cords, is deformed in such a way that the inner surface assumes a given shape. A previous solution for this problem, valid only when the curvature of the deformed inner surface is continuous everywhere, is extended to the more general case where the curvature possesses discontinuities. The method of matched asymptotic expansions is used to construct the solution in the neighborhood of a discontinuity, and to join it smoothly with the earlier solution, which is shown to remain valid away from the discontinuity.

The deformation described here occurs, for example, when a reinforced elastic tube is deformed due to an enclosed rigid mold or mandrel and an applied external pressure.

1. INTRODUCTION

The mathematical theory of finite elasticity is well established and several exact solutions to the governing equations have been obtained which describe a variety of deformations.† However, because the equations are highly non-linear, the number of such solutions is small, and remains so even with the simplifications which result from the assumption of a particular strain energy function, such as the Mooney–Rivlin[3, 4] or the neo-Hookean[5] form. Furthermore, exact solutions can usually be found only when the symmetry of the deformation is such that the equations can be reduced to a suitable form. Therefore, while these solutions are of great importance to the general theory, it cannot be expected that such solutions can be obtained for a problem which arises in practice. For this reason considerable effort has been devoted to the development of numerical and approximation techniques. Although the approximation techniques used so far assume a variety of forms, such as small deformations superposed on large deformations[6], perturbations of the strain energy function[7, 8], successive approximations [1, 2], or expansions in terms of a geometrical parameter[9], they are essentially similar in that they are in general all regular perturbation expansions. In this paper a singular perturbation technique is used to obtain the solution to a problem which has been previously solved for a restricted case by the use of regular perturbations[10].

The problem discussed here generalizes an earlier study of cylindrical deformations of an infinitely long elastic tube initially of circular cross section with inner radius r_1 and uniform thickness h_0. The tube is reinforced on its outside surface by a two parameter family of inextensible cords making constant angles $\pm\alpha$ with the generators of the surface. It is

† See for example [1, 2].

deformed under the action of a uniform pressure P^* on its outer surface and normal compressive surface forces on its inner surface in such a way that the cross section of the inner surface becomes a given closed curve c. The deformation can be realized physically by inserting a frictionless, rigid mold of given cross section into a circular cylindrical tube and applying a uniform external pressure.

A cross section of the deformed and undeformed configurations is shown in Fig. 1.

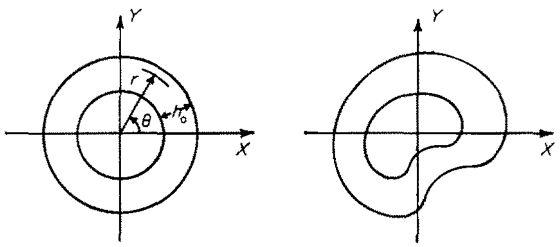

Fig. 1. Cross section of the cylinder.
(a) undeformed (b) deformed.

It was found that the solution at any point in the deformed body depends on the curvature of the inner surface at that point. If the curvature κ is discontinuous at a point, the solution obtained in[10] is also discontinuous at that point and the regular perturbation method cannot be applied. The case of a discontinuous curvature is examined in this paper. It is shown that the solution obtained in[10] still applies away from the point of discontinuity in κ, and a singular perturbation technique is used to develop a solution valid around the point of discontinuity which matches, away from it, the solution given in[10].

In Section 2 of this paper the problem is formulated and the solution for continuous curvature is summarized. This section is considerably abbreviated and the reader is referred to[10] for the details and further discussion.

The asymptotic expansion for the solution near a discontinuity is obtained in Section 3 and equations and boundary conditions which determine the deformation in this region are derived. It is shown in Section 4 that the problem reduces to the solution of the biharmonic equation on an infinite strip if there is no extension of the tube in the axial direction. This problem is solved by means of an eigenvalue expansion. The case where the tube is extended in the axial direction is solved in Section 5.

2. FORMULATION OF THE PROBLEM

Let (x, y, z) denote an orthogonal Cartesian coordinate system with the z-axis coinciding with the axis of the undeformed cylinder and (r, θ, z) denote the corresponding cylindrical coordinate system with $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. A point in the undeformed body with cylindrical coordinates $(r_1 + h_0 t, \theta, z)$ is displaced under the deformation to the point with Cartesian coordinates $(r_1 X(t, \theta), r_1 Y(t, \theta), lz)$ where the functions $X(t, \theta)$, $Y(t, \theta)$ and the constant extension ratio l are to be determined.

For an elastic, incompressible, homogeneous and isotropic material possessing a strain energy function of the neo-Hookean form† $W = C(I_1 - 3)$ suggested by Rivlin[5], the

† Since the deformations considered are plane, the analysis is also valid, with minor modifications, for the more general Mooney–Rivlin material.

governing equations are

$$
\begin{aligned}
l(X_t Y_\theta - X_\theta Y_t) &= \varepsilon(1+\varepsilon t)\\
(1+\varepsilon t)^2(X_\theta X_{tt} + Y_\theta Y_{tt}) + \varepsilon(1+\varepsilon t)(X_t X_\theta + Y_t Y_\theta)&\\
+ \varepsilon^2(X_\theta X_{\theta\theta} + Y_\theta Y_{\theta\theta}) + \varepsilon^2(1+\varepsilon t)^2 Q_\theta &= 0 \qquad (2.1)\\
(1+\varepsilon t)^2(X_t X_{tt} + Y_t Y_{tt}) + \varepsilon(1+\varepsilon t)(X_t^2 + Y_t^2)&\\
+ \varepsilon^2(X_t X_{\theta\theta} + Y_t Y_{\theta\theta}) + \varepsilon^2(1+\varepsilon t)^2 Q_t &= 0,
\end{aligned}
$$

where $2CQ(t, \theta)$ denotes the hydrostatic pressure in the material and

$$\varepsilon = h_0/r_1. \qquad (2.2)$$

The boundary conditions are

$$X_\theta^2 + Y_\theta^2 = l_0^2(1+\varepsilon)^2, \qquad t = 1 \qquad (2.3)$$

$$X_t X_\theta + Y_t Y_\theta = 0, \qquad t = 0 \qquad (2.4)$$

$$l_0^3(1+\varepsilon)^3\left[\frac{1}{l^2 l_0^2} + Q + P\right] = T(X_{\theta\theta} Y_\theta - X_\theta Y_{\theta\theta}), \qquad t = 1 \qquad (2.5)$$

$$\frac{1}{l l_0}(X_t X_\theta + Y_t Y_\theta) = \varepsilon T_\theta, \qquad t = 1 \qquad (2.6)$$

$$X(0, \theta) = L\zeta(s), \qquad Y(0, \theta) = L\xi(s), \qquad 0 \le s \le 1 \qquad (2.7)$$

where

$$l_0 = (\sin\alpha)^{-1}\sqrt{1 - l^2\cos^2\alpha} \qquad (2.8)$$

and

$$T = T^*/2Cr_1 \qquad (2.9)$$

is the non-dimensional tension per unit length applied across a generator of the reinforcing material. The non-dimensional applied pressure is defined by $P = P^*/2C$. Equation (2.3) expresses the inextensibility of the cords, (2.4) the absence of shear forces on the inner surface, and (2.5), (2.6) the equilibrium of the reinforcing material. Equation (2.7) specifies the shape of the inner surface in terms of the functions $\zeta(s)$ and $\xi(s)$, which satisfy

$$\zeta'^2 + \xi'^2 = 1 \qquad (2.10)$$

and

$$\zeta(0) = \lambda/L, \qquad \xi(0) = 0, \qquad \zeta'(0) = 0 \qquad (2.11)$$

where λ is a non-dimensional constant. The restrictions (2.11) fix the position of the deformed body.

Details of the derivation of the governing equations (2.1) and the boundary conditions (2.3–2.7) may be found in[10].

The problem formulated above was solved in[10] by means of a perturbation solution applicable when ε, the ratio of the thickness h_0 of the undeformed tube to its inside radius r_1, is small compared to unity. The dependent variables X, Y and Q, as well as the independent variable θ, were expanded in a series in ε with coefficients which are functions of t and a new non-dimensional independent variable ϕ such that θ is an increasing function of ϕ and $\theta = 0$ when $\phi = 0$. The solutions obtained are

$$X = L\zeta(s) + \varepsilon \frac{t}{ll_0} \xi'(s) + \varepsilon^2 \left\{ \frac{1}{2l^2 l_0^2} t(t-2)[ll_0 - \kappa(s)]\xi'(s) + \frac{T_0}{2L} t^2 \zeta'(s)\kappa'(s) \right\} + 0(\varepsilon^3)$$

$$Y = L\xi(s) - \varepsilon \frac{t}{ll_0} \zeta'(s) + \varepsilon^2 \left\{ \frac{-1}{2l^2 l_0^2} t(t-2)[ll_0 - \kappa(s)]\zeta'(s) + \frac{T_0}{2L} t^2 \xi'(s)\kappa'(s) \right\} + 0(\varepsilon^3) \quad (2.12)$$

$$Q = -P - \frac{1}{l^2 l_0^2} - T_0 \kappa(s) + \varepsilon \left\{ \left[\frac{1 + l^2 l_0^4}{l^3 l_0^3} \kappa(s) - \frac{2}{l^2 l_0^2} \right](t-1) + \left[\frac{T_0}{ll_0} \kappa(0) - T_1(0) \right] \kappa(s) \right\} + 0(\varepsilon^2)$$

where

$$s = \frac{l_0}{L} \phi \quad (2.13)$$

and

$$\theta = \phi + \varepsilon a_1(\phi) + \varepsilon^2 a_2(\phi) + \cdots \quad (2.14)$$

$$\phi = \theta - \varepsilon a_1(\theta) + \varepsilon^2 [a_1(\theta) a_1'(\theta) - a_2(\theta)] + \cdots.$$

The curvature of c is denoted by $\kappa(\eta)/r_1$, where

$$\kappa(\eta) = \frac{1}{L} [\zeta'(\eta)\xi''(\eta) - \xi'(\eta)\zeta''(\eta)]. \quad (2.15)$$

The a_i are given by

$$a_1(\phi) = \frac{L}{ll_0^2} \int_0^s [\kappa(\tau) - ll_0] \, d\tau$$

$$a_2(\phi) = \frac{T_0}{2Ll_0} [\kappa'(s) - \kappa'(0)] + \frac{L}{2l^2 l_0^3} \int_0^s [\kappa^2(\tau) - 3ll_0 \kappa(\tau) + 2l^2 l_0^2] \, d\tau. \quad (2.16)$$

The tension $T(\phi)$ is

$$T = T(0) + \varepsilon \frac{T_0}{ll_0} [\kappa(s) - \kappa(0)] + \varepsilon^2 \frac{2 + 2l^2 l_0^4 - T_0 l^2 l_0^2 [ll_0 + 2\kappa(0)] + 2T_1(0) l^3 l_0^3}{2l^4 l_0^4} [\kappa(s) - \kappa(0)] + 0(\varepsilon^3) \quad (2.17)$$

where $T(0) = T_0 + \varepsilon T_1(0) + \varepsilon^2 T_2(0) + \ldots$ is the value of T at $\phi = 0$. Since the cords can support a load of any order, the tension must be specified arbitrarily at some point, and this is done by giving $T(0)$. In the remainder of this paper it will be assumed that $T_0 = 0$, so that the loading on the cords vanishes as $\varepsilon \to 0$. In this case (2.17) is replaced by

$$T = \varepsilon T_1(0) + \varepsilon^2 \left\{ T_2(0) + \left[\frac{1}{ll_0} T_1(0) + \frac{1 + l^2 l_0^4}{l^4 l_0^4} \right] [\kappa(s) - \kappa(0)] \right\}. \quad (2.18)$$

Expressions for the stresses and a discussion of these solutions can be found in [10].

3. THE CASE OF DISCONTINUOUS CURVATURE

The solutions shown in the previous section are discontinuous at those values of ϕ for which the curvature κ of the inner surface is discontinuous. Since it is clear physically that a discontinuity in the curvature of the prescribed shape of the inner surface cannot produce jumps in the displacements, stresses, etc. at that point, the expansion must be invalid in the neighborhood of discontinuities in κ. The reason for the breakdown of the solution is that the expansion used in[10] is equivalent to the assumption that derivatives with respect to θ are small compared to derivatives with respect to t. However, in the neighborhood of a discontinuity in κ the dependent variables change rapidly along the tube and can no longer be regarded as small compared to derivatives across the tube. It will be shown that these solutions still apply for values of θ bounded away from any discontinuity in curvature by constructing a local solution which describes the deformation in the neighborhood of the discontinuity and provides a smooth transition between the solutions on either side of the discontinuity.

For simplicity, the coordinate system is chosen so that the discontinuity in curvature occurs at $\theta = \phi = 0$ and $\zeta(0) = \xi(0) = \zeta'(0) = 0$ (c.f. (2.11)). In order to obtain a solution near $\theta = 0$ in the limit $\varepsilon \to 0$ new independent variables must be introduced. These variables, defined by

$$\eta = \theta/\varepsilon, \qquad \tau = t, \tag{3.1}$$

have the property that the limit $\varepsilon \to 0$, η, τ fixed, implies $\theta \to 0$, and so are the appropriate variables to describe the solution in this region†.

The dependent variables expressed as functions of η and τ are denoted by

$$\left.\begin{aligned} \hat{X}(\tau, \eta) &= X(t, \theta) \\ \hat{Y}(\tau, \eta) &= Y(t, \theta) \\ \hat{Q}(\tau, \eta) &= Q(t, \theta) \\ \hat{T}(\eta) &= T(\theta). \end{aligned}\right\} \tag{3.2}$$

In terms of these variables the governing equations take the form

$$\begin{gathered} l(\hat{X}_\tau \hat{Y}_\eta - \hat{Y}_\tau \hat{X}_\eta) = \varepsilon^2(1 + \varepsilon\tau) \\ (1 + \varepsilon\tau)^2(\hat{X}_\eta \hat{X}_{\tau\tau} + \hat{Y}_\eta \hat{Y}_{\tau\tau}) + \varepsilon(1 + \varepsilon\tau)(\hat{X}_\tau \hat{X}_\eta + \hat{Y}_\tau \hat{Y}_\eta) \\ + (\hat{X}_\eta \hat{X}_{\eta\eta} + \hat{Y}_\eta \hat{Y}_{\eta\eta}) + \varepsilon^2(1 + \varepsilon\tau)^2 \hat{Q}_\eta = 0 \\ (1 + \varepsilon\tau)^2(\hat{X}_\tau \hat{X}_{\tau\tau} + \hat{Y}_\tau \hat{Y}_{\tau\tau}) + \varepsilon(1 + \varepsilon\tau)(\hat{X}_\tau^2 + \hat{Y}_\tau^2) + (\hat{X}_\tau \hat{X}_{\eta\eta} \\ + \hat{Y}_\tau \hat{Y}_{\eta\eta}) + \varepsilon^2(1 + \varepsilon\tau)^2 \hat{Q}_\tau = 0 \end{gathered} \tag{3.3}$$

and the boundary conditions are

$$\begin{aligned} &\hat{X}_\tau \hat{X}_\eta + \hat{Y}_\tau \hat{Y}_\eta = 0 && \tau = 0 \\ &\hat{X}_\eta^2 + \hat{Y}_\eta^2 = \varepsilon^2 l_0^2(1 + \varepsilon)^2 && \tau = 1 \\ &l_0^3\varepsilon^3(1 + \varepsilon)^3\left(\frac{1}{l^2 l_0^2} + \hat{Q} + P\right) = \hat{T}(\hat{X}_{\eta\eta} \hat{Y}_\eta - \hat{X}_\eta \hat{Y}_{\eta\eta}) && \tau = 1 \\ &\varepsilon\frac{d\hat{T}}{d\eta} = \frac{1}{l l_0}(\hat{X}_\tau \hat{X}_\eta + \hat{Y}_\tau \hat{Y}_\eta) && \tau = 1 \\ &\hat{X}(0, \eta) = L\zeta(s), \qquad \hat{Y}(0, \eta) = L\xi(s). \end{aligned} \tag{3.4}$$

† The ε in (3·1) may be replaced by an arbitrary function $f(\varepsilon)$ with the property $f(\varepsilon) \to 0$, $\varepsilon \to 0$. It is then found in the course of the analysis that the only possible choice is $f(\varepsilon) = \varepsilon$. For simplicity, this choice is made at the outset.

Additional conditions are provided by the requirement that the solution as $\eta \to +\infty$ or $\eta \to -\infty$ must match the solution obtained in [10] as $\theta \to 0^+$ or $\theta \to 0^-$, respectively. Therefore, in order to determine these conditions, as well as the nature of the asymptotic series describing the solution in the transition layer, it is necessary to study the behavior of the solutions (2.12) as $\theta \to 0$. This is done by rewriting these solutions in terms of η and τ and expanding the result as a series in ε, since, as already pointed out, the limit $\varepsilon \to 0$, η fixed, implies $\theta \to 0$.

It is necessary, therefore, to determine $\phi(\eta)$. Combining $(2.14)_1$ and $(3.1)_1$ and expanding in ε yields

$$\phi = \varepsilon\eta - \varepsilon^2\left(\frac{\kappa_0^\pm}{ll_0} - 1\right)\eta \tag{3.5}$$

where κ_0^+, κ_0^- are the limits of κ as $\theta \to 0^+$ and $\theta \to 0^-$ respectively, and κ_0^+ or κ_0^- is used in (3.5) for $\eta > 0$ and $\eta < 0$, respectively. Substituting (3.5) into the solutions (2.12) and expanding in ε gives

$$\begin{aligned} X &\sim \varepsilon\frac{\tau}{ll_0} + \varepsilon^2\left\{\frac{1}{ll_0}\left(\frac{\tau^2}{2} - \tau\right)\left(1 - \frac{\kappa_0^\pm}{ll_0}\right) - \frac{\kappa_0^\pm l_0^2}{2}\eta^2\right\} \\ Y &\sim \varepsilon l_0\eta + \varepsilon^2\left\{\frac{1}{l}\kappa_0^\pm\eta\tau + l_0\left(1 - \frac{\kappa_0^\pm}{ll_0}\right)\eta\right\} \\ Q &\sim -P - \frac{1}{l^2l_0^2} + \varepsilon\left\{\left[\frac{1 + l^2l_0^4}{l^3l_0^3}\kappa_0^\pm - \frac{2}{l^2l_0^2}\right](\tau - 1) - T_1(0)\kappa_0^\pm\right\} \end{aligned} \tag{3.6}$$

where again κ_0^+ and κ_0^- are used for $\eta > 0$ and $\eta < 0$, respectively. Clearly, the leading term in each of these expressions is the same as $\eta \to 0^+$ or $\eta \to 0^-$, the discontinuity appearing only in terms of ε^2 or higher in X and Y and in ε or higher in Q. From this it follows that the appropriate expansion for the transition layer is

$$\begin{aligned} \hat{X}(\tau, \eta; \varepsilon) &\sim \varepsilon\frac{\tau}{ll_0} + \varepsilon^2 U(\tau, \eta) + \ldots \\ \hat{Y}(\tau, \eta; \varepsilon) &\sim \varepsilon l_0\eta + \varepsilon^2 V(\tau, \eta) + \ldots \\ \hat{Q}(\tau, \eta; \varepsilon) &\sim -P - \frac{1}{l^2l_0^2} + \varepsilon W(\tau, \eta) + \ldots \end{aligned} \tag{3.7}$$

and that the matching conditions are

$$\begin{aligned} U &\to \frac{1}{ll_0}\left(\frac{\tau^2}{2} - \tau\right)\left(1 - \frac{\kappa_0^\pm}{ll_0}\right) - \frac{\kappa_0^\pm l_0^2}{2}\eta^2 \\ V &\to \frac{1}{l}\kappa_0^\pm\eta\tau + l_0\left(1 - \frac{\kappa_0^\pm}{ll_0}\right)\eta \\ W &\to \left[\frac{1 + l^2l_0^4}{l^3l_0^3}\kappa_0^\pm - \frac{2}{l^2l_0^2}\right](\tau - 1) - T_1(0)\kappa_0^\pm \end{aligned} \tag{3.8}$$

as $\eta \to \pm\infty$, where κ_0^+ is used for $\eta \to +\infty$ and κ_0^- for $\eta \to -\infty$.

Equations for U, V and W are obtained by substituting (3.7) into (3.3) and retaining the leading terms. This gives

$$\begin{aligned} ll_0 U_\tau + \frac{1}{l_0} V_\eta &= \tau \\ l_0(V_{\tau\tau} + V_{\eta\eta}) + W_\eta &= 0 \\ \frac{1}{ll_0}(U_{\tau\tau} + U_{\eta\eta}) + \frac{1}{(ll_0)^2} + W_\tau &= 0. \end{aligned} \tag{3.9}$$

The boundary conditions are derived by substituting the expansions (3.7) into (3.4). Assuming the tension $\hat{T}(\eta)$ can be expanded in the form

$$\hat{T}(\eta) = \varepsilon Z_1(\eta) + \varepsilon^2 Z_2(\eta) + \dots \tag{3.10}$$

the boundary conditions are

$$\frac{1}{ll_0} U_\eta + l_0 V_\tau = 0 \qquad \tau = 0 \tag{3.11}$$

$$V_\eta = l_0 \qquad \tau = 1 \tag{3.12}$$

$$l_0^2 W = Z_1 U_{\eta\eta} \qquad \tau = 1 \tag{3.13}$$

$$\frac{dZ_1}{d\eta} = 0 \tag{3.14}$$

$$\frac{dZ_2}{d\eta} = \frac{1}{(ll_0)^2}(U_\eta + ll_0^2 V_\tau) \qquad \tau = 1 \tag{3.15}$$

$$U(\eta, 0) = -\tfrac{1}{2}\kappa_0^\pm l_0^2 \eta^2 \ \eta \gtrless 0, \qquad \tau = 0. \tag{3.16}$$

It follows from (3.14) that

$$Z_1 = \text{constant} = T_1(0).$$

Therefore, the cord tension is constant throughout to first order in ε.

Introducing the change of variables

$$\hat{\eta} = l_0 \sqrt{l}\, \eta \tag{3.17}$$

and

$$U = \frac{1}{ll_0}\left(\Phi + \frac{\tau^2}{2}\right), \tag{3.18}$$

it can be seen that equation $(3.9)_1$ implies the existence of a stream function $\Psi(\tau, \hat{\eta})$ such that

$$V = -\Psi_\tau, \qquad \Phi = \sqrt{l}\, \Psi_\eta. \tag{3.19}$$

Substituting (3.19) into $(3.9)_{2,3}$ and eliminating W gives an equation for Ψ:

$$\Psi_{\tau\tau\tau\tau} + \lambda \Psi_{\tau\tau\eta\eta} + \Psi_{\eta\eta\eta\eta} \tag{3.20}$$

where

$$\lambda = ll_0^2 + \frac{1}{ll_0^2}. \tag{3.21}$$

Boundary conditions for Ψ are obtained from (3.11)–(3.13) and (3.16):

$$\frac{1}{ll_0^2}\Psi_{\eta\eta} - \Psi_{\tau\tau} = 0 \qquad \tau = 0 \tag{3.22}$$

$$\Psi_{\eta\tau} = -1/\sqrt{l} \qquad \tau = 1 \tag{3.23}$$

$$\Psi_{\tau\tau\tau} + ll_0\Psi_{\eta\eta\tau} = T_1(0)\frac{l}{l_0}\Psi_{\eta\eta\eta\eta} \qquad \tau = 1 \tag{3.24}$$

$$\left.\begin{aligned}\Psi_\eta &= -\tfrac{1}{2}\kappa_0^+ \frac{l_0}{\sqrt{l}}\hat{\eta}^2\eta > 0, \qquad \tau = 0\\ \Psi_\theta &= -\tfrac{1}{2}\kappa_0^- \frac{l_0}{\sqrt{l}}\hat{\eta}^2\eta < 0, \qquad \tau = 0\end{aligned}\right\} \tag{3.25}$$

where $(3.9)_2$ was used to eliminate W in obtaining (3.24) from (3.13).

The matching conditions $(3.8)_{1,2}$ can be written in terms of the stream function as

$$\Psi \to -\frac{\tau^2\hat{\eta}}{2}\frac{\kappa_0^\pm}{l_0 l^{3/2}} - \frac{1}{\sqrt{l}}\hat{\eta}\tau\left(1 - \frac{\kappa_0^\pm}{ll_0}\right) - \tfrac{1}{6}\kappa_0^\pm\frac{l_0}{\sqrt{l}}\hat{\eta}^3 \tag{3.26}$$

for $\eta \to \pm\infty$. The integration constant which results when (3.26) is obtained from the expressions for Ψ_η and Ψ_τ (i.e. $(3.8)_{1,2}$) is zero since there is no transition layer when $\kappa_0^+ = \kappa_0^-$ and Ψ must reduce to

$$\Psi = -\frac{\tau^2\hat{\eta}}{2}\frac{\kappa_0}{l_0 l^{3/2}} - \frac{1}{\sqrt{l}}\hat{\eta}\tau\left(1 - \frac{\kappa_0}{ll_0}\right) - \tfrac{1}{6}\kappa_0\frac{l_0}{\sqrt{l}}\hat{\eta}^3$$

where $\kappa_0 = \kappa_0^+ = \kappa_0^-$.

Equation (3.20), the boundary conditions (3.22)–(3.25), and the matching condition (3.26) determine the function Ψ, defined on the infinite strip $-\infty < \hat{\eta} < +\infty$, $0 \le \tau \le 1$. Once Ψ is obtained, Z_2 can be found from (3.15) and W can be found from

$$W = -\frac{\sqrt{l}}{(ll_0)^2}\left(\Psi\hat{\eta}\tau + \frac{1}{\sqrt{l}}\right) - \frac{1}{\sqrt{l}}\int_1^\tau \Psi\hat{\eta}\hat{\eta}\hat{\eta}\,d\tau - \frac{2}{(ll_0)^2}(\tau - 1) + \frac{\sqrt{l}}{l_0}T_1(0)\Psi\hat{\eta}\hat{\eta}\hat{\eta}\,|_{\tau=1}. \tag{3.27}$$

Equation (3.27) was derived by integrating $(3.9)_3$ and applying the boundary condition (3.13).

The stress in the transition layer can also be obtained from these solutions. For example, Σ_{11}, the normal stress on a surface r = constant, is given in[10] as

$$\frac{\Sigma_{11}}{2C} = \frac{(1 + \varepsilon t)^2}{l^2(x_\theta^2 + y_\theta^2)} + Q.$$

Introducing the variables (η, τ) and the expansion (3.7) and expanding the results in ε shows that in the transition layer

$$\frac{\Sigma_{11}}{2C} = -P + \varepsilon\left\{W - \frac{2}{l^2 l_0^3} V_\eta + \frac{2}{l^2 l_0^2}\tau\right\}. \tag{3.28}$$

4. THE TRANSITION LAYER SOLUTION

In this section the problem for Ψ formulated above will be simplified by introducing two restrictions. The first is that $l = 1$. It then follows from (2.8) that $l_0 = 1$, and from (3.21) that $\lambda = 2$, so that equation (3.20) for Ψ becomes the biharmonic equation.† The same result is obtained under the weaker condition that $l - 1$ is of order ε. Physically this means that there is no movement ($l = 1$) or very slight movement ($l - 1 = 0(\varepsilon)$) in the axial direction. When $l = 1$ the length L of the curve c, the cross section of the inside surface of the deformed cylinder, is given by $L = 2\pi + \varepsilon^2\beta$ where β depends on the shape of c and is given in[10], and 2π is the nondimensional circumference of the undeformed cylinder. When $l - 1 = 0(\varepsilon)$, then $L - 2\pi = 0(\varepsilon)$. Therefore, if the length of the inner surface differs in the undeformed and deformed states by at most $0(\varepsilon)$, the transition layer at a discontinuity in curvature is described by the biharmonic equation.

The second restriction is that $T_1(0) = 0$. It has already been pointed out that the tension is constant to $0(\varepsilon)$, and that this constant, $T_1(0)$, represents an applied load. It is therefore assumed that any applied load is $0(\varepsilon^2)$, which is the same order as the tension induced in the cords by the elastic material as a result of the deformation.

Solutions to the biharmonic equation which satisfy the matching conditions at $\eta = \pm\infty$ and the boundary conditions at $\tau = 0, 1$ are given by

$$\Psi = -\eta\left(\frac{\tau^2}{2} - \tau\right)\kappa_0^+ - \tfrac{1}{6}\eta^3\kappa_0^+ - \eta\tau + \sum_{k=0}^{\infty}(a_k + \eta b_k)\sin(\delta_k\tau)e^{-\delta_k\eta} \tag{4.1}$$

for $\eta \geq 0$, and

$$\Psi = -\eta\left(\frac{\tau^2}{2} - \tau\right)\kappa_0^- - \tfrac{1}{6}\eta^3\kappa_0^- - \eta\tau + \sum_{k=0}^{\infty}(A_k + \eta B_k)\sin(\delta_k\tau)e^{\delta_k\eta} \tag{4.2}$$

for $\eta \leq 0$, where

$$\delta_k = (2k+1)\frac{\pi}{2}. \tag{4.3}$$

The unknown constants a_k, b_k, A_k, B_k are determined by matching Ψ, Ψ_η, $\Psi_{\eta\eta}$ and $\Psi_{\eta\eta\eta}$, obtained from (4.1) for $\eta \geq 0$ and (4.2) for $\eta \leq 0$, at their common boundary $\eta = 0$. Since Ψ satisfies a fourth order equation in η, it follows that all higher derivatives in η will be continuous at $\eta = 0$. These matching conditions give

$$\begin{aligned}
&\Sigma a_k \sin\delta_k\tau = \Sigma A_k \sin\delta_k\tau \\
&\Sigma(-a_k\delta_k + b_k)\sin\delta_k\tau = \Sigma(A_k\delta_k + B_k)\sin\delta_k\tau + \left(\frac{\tau^2}{2} - \tau\right)[\kappa_0] \\
&\Sigma(a_k\delta_k^2 - 2b_k\delta_k)\sin\delta_k\tau = \Sigma(A_k\delta_k^2 + 2B_k\delta_k)\sin\delta_k\tau \\
&\Sigma(-a_k\delta_k^3 + 3b_k\delta_k^2)\sin\delta_k\tau = \Sigma(A_k\delta_k^3 + 3B_k\delta_k^2)\sin\delta_k\tau + [\kappa_0]
\end{aligned} \tag{4.4}$$

† The biharmonic equation is also obtained for values of l other than $l=1$, determined by solving the equation $\lambda=2$. The subsequent analysis holds with minor modification for deformations with these particular values of the extension ratio. These special cases will not be specifically investigated here. In the following analysis, it is assumed that $l=l_0=1$, so that $\hat{\eta}=\eta$.

where $[\kappa_0] = \kappa_0^+ - \kappa_0^-$ is the jump in the curvature at $\eta = 0$. The first and third of these equations give $A_k = a_k$, $B_k = -b_k$ and the remaining two equations give

$$\begin{aligned} a_k &= A_k = \frac{2}{\delta_k^4}[\kappa_0] \\ b_k &= -B_k = \frac{1}{\delta_k^3}[\kappa_0]. \end{aligned} \tag{4.5}$$

If these results are substituted into $(4.4)_2$ and $(4.4)_4$ the series can be summed, thereby verifying the matching conditions.

The displacements U and V are obtained simply by differentiating the stream function and W is determined from (3.27), which gives for $\eta > 0$

$$W = 2(\tau - 1)(\kappa_0^+ - 1) + [\kappa_0] \sum_{k=0}^{\infty} \frac{2}{\delta_k^2} \cos \delta_k \tau e^{-\delta_k \eta} \tag{4.6}$$

and for $\eta < 0$

$$W = 2(\tau - 1)(\kappa_0^- - 1) - [\kappa_0] \sum_{k=0}^{\infty} \frac{2}{\delta_k^2} \cos \delta_k \tau e^{\delta_k \eta}. \tag{4.7}$$

It can easily be seen that W is continuous at $\eta = 0$. In particular, from either (4.6) or (4.7)

$$W(0, \tau) = (\kappa_0^+ + \kappa_0^- - 2)(\tau - 1). \tag{4.8}$$

The tension in the cords, $Z_2(\eta)$, can be found by substituting the above solution into (3.15). This gives, for $\eta \leq 0$

$$\begin{aligned} Z_2 &= \int_{-\infty}^{\eta} (\Psi_{\eta\eta} - \Psi_{\tau\tau})_{\tau=0}\, d\eta + Z_2(-\infty) \\ &= [\kappa_0] \sum_{k=0}^{\infty} \sin \delta_k \left(\frac{4}{\delta_k^3} - \frac{2}{\delta_k^2}\eta\right) e^{\delta_k \eta} + Z_2(-\infty) \end{aligned} \tag{4.9}$$

and for $\eta \geq 0$

$$\begin{aligned} Z_2 &= \int_0^{\eta} (\Psi_{\eta\eta} - \Psi_{\tau\tau})_{\tau=0}\, d\eta + Z_2(0) \\ &= -[\kappa_0] \sum_{k=0}^{\infty} \sin \delta_k \left(\frac{4}{\delta_k^3} + \frac{2}{\delta_k^2}\eta\right) e^{-\delta_k \eta} + 2[\kappa_0] + Z_2(-\infty) \end{aligned} \tag{4.10}$$

where $Z_2(0) = [\kappa_0] + Z_2(-\infty)$ was found from (4.9). From (4.10) it follows that

$$[T] = \varepsilon^2\{Z_2(+\infty) - Z_2(-\infty)\} = 2\varepsilon^2[\kappa_0]; \tag{4.11}$$

that is, the jump in tension, $[T]$, is equal to $2\varepsilon^2$ times the jump in the curvature. These results for $Z_2(\eta)$ are shown in Fig. 2.

If the tension at one end of the transition layer is specified (e.g. $T(-\infty)$) the variation in tension through the layer, and in particular, the total change across the transition layer is determined. However, while the analysis in[10] does not describe the tension in the transition layer, it can be shown from those results that $[T] = \lambda[\kappa_0]$, where λ = constant. For the case considered here ($T_0 = T_1(0) = 0$, $l = 1$), $\lambda = 2\varepsilon^2$, consistent with the result (4.11).

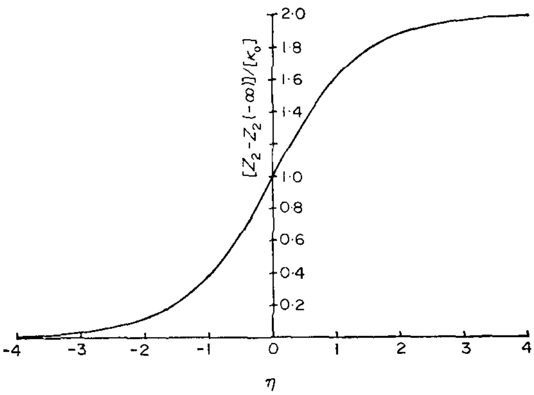

Fig. 2. Variation of cord tension through the transition layer.

The stress Σ_{11} in the transition layer is found by substituting the solution into (3.28). This gives

$$\frac{\Sigma_{11}}{2C} + P = -\varepsilon[\kappa_0]2\eta \sum_{k=0}^{\infty} \frac{1}{\delta_k} \cos \delta_k \tau e^{-\delta_k \eta} \tag{4.12}$$

for $\eta > 0$, and

$$\frac{\Sigma_{11}}{2C} + P = -\varepsilon[\kappa_0]2\eta \sum_{k=0}^{\infty} \frac{1}{\delta_k} \cos \delta_k \tau e^{\delta_k \eta} \tag{4.13}$$

for $\eta < 0$. At $\tau = 1$, $\Sigma_{11}/2C = -P + 0(\varepsilon^2)$, and at $\tau = 0$

$$\frac{\Sigma_{11}}{2C} + P = \varepsilon[\kappa_0]\frac{\pi}{2}\eta \, ln \tanh \pi|\eta|. \tag{4.14}$$

This result is illustrated in Fig. 3.

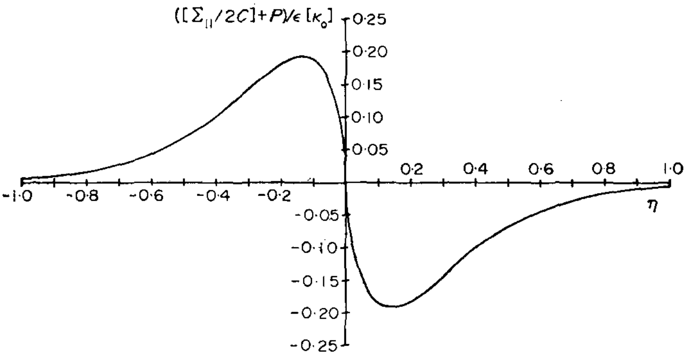

Fig. 3. Variation of normal stress on the inner surface $\tau=0$.

At $\tau = 0$, Σ_{11} gives the stress applied to the inner surface of the cylinder. In[10] it is found that $\Sigma_{11} + P = 0(\varepsilon^2)$ at $\tau = 0$ for the case considered here ($l = 1$, $T_0 = T_1(0) = 0$). Equation (4.14) shows that this quantity is an order of magnitude larger in the transition layer.

Since the stress Σ_{11} at $\tau = 0$ cannot be tensile, (4.14) requires

$$P \geq \varepsilon[\kappa_0] \frac{\pi}{2} \max\{\eta \, ln \tanh \pi|\eta|\} = 0.194\, \varepsilon[\kappa_0] \tag{4.15}$$

which gives a lower bound on the magnitude of P. Physically, if (4.15) is not satisfied the cylinder separates from the enclosed rigid mold which causes the deformation. In particular, if $P = 0$, $\Sigma_{11} < 0$ for $\eta < 0$ and $\Sigma_{11} > 0$ for $\eta > 0$, so that the cylinder separates for $\eta < 0$. For this case the boundary condition at $\tau = 0$, $\eta < 0$, which specifies the shape of the inner surface (i.e. equation 3.25_2) is replaced by the condition that $\Sigma_{11} = 0$. This can be expressed as

$$\Psi_{\tau\tau\tau} + 3\Psi_{\eta\eta\tau} = 0, \qquad \tau = 0, \qquad \eta < 0. \tag{4.16}$$

The solution to the biharmonic equation with (4.16) replacing $(3.25)_2$ is much more complicated than the present case and will be discussed in a subsequent paper.

5. SOLUTION FOR FINITE AXIAL EXTENSION

If the axial extension parameter l is not assumed to be near unity, the transition layer is governed by equation (3.20). Solutions to this equation which satisfy the boundary and matching conditions (3.22)–(3.26) are (for simplicity the $\hat{}$ is eliminated from $\hat{\eta}$)

$$\Psi = -\eta\left(\frac{\tau^2}{2} - \tau\right)\frac{\kappa_0^+}{l_0 l^{3/2}} - \frac{\eta\tau}{\sqrt{l}} - \tfrac{1}{6}\eta^3\kappa_0^+ \frac{l_0}{\sqrt{l}} + \sum_{k=0}^{\infty}(a_k e^{-\mu_k\eta} + b_k e^{-\nu_k\eta})\sin\delta_k\tau \tag{5.1}$$

for $\eta \geq 0$ and

$$\Psi = -\eta\left(\frac{\tau^2}{2} - \tau\right)\frac{\kappa_0^-}{l_0 l^{3/2}} - \frac{\eta\tau}{\sqrt{l}} - \tfrac{1}{6}\eta^3\kappa_0^- \frac{l_0}{\sqrt{l}} + \sum_{k=0}^{\infty}(A_k e^{\mu_k\eta} + B_k e^{\nu_k\eta})\sin\delta_k\tau \tag{5.2}$$

for $\eta \leq 0$, where

$$\begin{aligned}
\delta_k &= (2k+1)\frac{\pi}{2} \\
\mu_k &= \left[\frac{\lambda}{2} + \sqrt{\left(\frac{\lambda}{2}\right)^2 - 1}\right]^{1/2}\delta_k = r_1\delta_k \\
\nu_k &= \left[\frac{\lambda}{2} - \sqrt{\left(\frac{\lambda}{2}\right)^2 - 1}\right]^{1/2}\delta_k = r_2\delta_k .
\end{aligned} \tag{5.3}$$

The constants a_k, b_k, A_k, B_k are found by matching Ψ, Ψ_η, $\Psi_{\eta\eta}$ and $\Psi_{\eta\eta\eta}$ at $\eta = 0$, which gives the equations

$$\begin{aligned}
\Sigma(a_k + b_k)\sin\delta_k\tau &= \Sigma(A_k + B_k)\sin\delta_k\tau \\
-\Sigma(a_k\mu_k + b_k\nu_k)\sin\delta_k\tau &= \Sigma(A_k\mu_k + B_k\nu_k)\sin\delta_k\tau + \frac{1}{l_0 l^{3/2}}\left(\frac{\tau^2}{2} - \tau\right)[\kappa_0] \\
\Sigma(a_k\mu_k^2 + b_k\nu_k^2)\sin\delta_k\tau &= \Sigma(A_k\mu_k^2 + B_k\nu_k^2)\sin\delta_k\tau \\
-\Sigma(a_k\mu_k^3 + b_k\nu_k^3)\sin\delta_k\tau &= \Sigma(A_k\mu_k^3 + B_k\nu_k^3)\sin\delta_k\tau + \frac{l_0}{\sqrt{l}}[\kappa_0].
\end{aligned} \tag{5.4}$$

The first and third of these equations give $a_k = A_k$, $b_k = B_k$, and the second and fourth give

$$A_k = a_k = \frac{[\kappa_0]}{\delta_k^4 r_1(r_2^2 - r_1^2)}\left(\frac{l_0}{\sqrt{l}} + \frac{r_2^2}{l_0 l^{3/2}}\right)$$
$$B_k = b_k = \frac{[\kappa_0]}{\delta_k^4 r_2(r_1^2 - r_2^2)}\left(\frac{l_0}{\sqrt{l}} + \frac{r_1^2}{l_0 l^{3/2}}\right). \tag{5.5}$$

It can be seen from the form of equation (5.1) that the solution obtained in the previous section for $l = 1$ cannot be found from the results of this section by taking the limit $l \to 1$.

The cord tension is found by substituting (5.1) into (3.15). This gives, for $\eta \leq 0$,

$$Z_2 = \frac{1}{l_0^3 l^{5/2}} \sum_{k=0}^{\infty} \delta_k \sin \delta_k \left\{ A_k \frac{r_1^2 + l l_0^2}{r_1} e^{\mu_k \eta} + B_k \frac{r_2^2 + l l_0^2}{r^2} e^{\nu_k \eta} \right\} + Z_2(-\infty) \tag{5.6}$$

and for $\eta \geq 0$

$$Z_2 = \frac{1}{l_0^3 l^{5/2}} \sum_{k=0}^{\infty} \delta_k \sin \delta_k \left\{ A_k \frac{r_1^2 + l l_0^2}{r_1} (1 - e^{-\mu_k \eta}) + B_k \frac{r_2^2 + l l_0^2}{r_2} (1 - e^{-\nu_k \eta}) \right\} + Z_2(0). \tag{5.7}$$

When $\eta = 0$, the series in (5.6) can be summed, and it can be shown that

$$Z_2(0) = \tfrac{1}{2}[\kappa_0]\left\{\frac{1}{l_0^4 l^4} + \frac{1}{l^2}\right\} + Z_2(-\infty). \tag{5.8}$$

Substituting this result into (5.7), setting $\eta = +\infty$, and summing the resulting series gives

$$[Z_2] = Z_2(+\infty) - Z_2(-\infty) = [\kappa_0]\left\{\frac{1}{l_0^4 l^4} + \frac{1}{l^2}\right\}. \tag{5.9}$$

This agrees with the result obtained in[10] for the jump in cord tension across a discontinuity in curvature for the case when $T_0 = 0$, $T_1(0) = 0$, $l \neq 1$.

The stress Σ_{11} can be determined from (3.28) and is given by

$$\frac{\Sigma_{11}}{2C} + P = \varepsilon\left\{\kappa_0^{\pm}(t-1)\left(\frac{l_0}{l} - \frac{1}{l^3 l_0^3}\right) \mp \frac{r_1}{\sqrt{l}}\left(\frac{1}{l l_0^2} + r_1^2\right)\delta_k^2 A_k e^{\mp \mu_k \eta} \cos \delta_k \tau \right.$$
$$\left. \mp \frac{r_2}{\sqrt{l}}\left(\frac{1}{l l_0^2} + r_2^2\right)\delta_k^2 B_k e^{\mp \nu_k \eta} \cos \delta_k \tau\right\} \tag{5.10}$$

where the upper sign applies for $\eta \geq 0$ and the lower sign for $\eta \leq 0$. As $\eta \to +\infty$ or $\eta \to -\infty$,

$$\frac{\Sigma_{11}}{2C} + P \to \varepsilon\left\{\kappa_0^{\pm}(t-1)\left(\frac{l_0}{l} - \frac{1}{l^3 l_0^3}\right)\right\} \tag{5.11}$$

which are the values of the stress on either side of the discontinuity, as given in[10].

When $\eta = 0$, the series in (5.10) can be summed, and the stress, either from the expression for $\eta \geq 0$ or for $\eta \leq 0$, reduces to

$$\frac{\Sigma_{11}(\tau, 0)}{2C} + P = \varepsilon\left\{\tfrac{1}{2}(\kappa_0^+ + \kappa_0^-)(\tau - 1)\left(\frac{l_0}{l} - \frac{1}{l^3 l_0^3}\right)\right\}.$$

It can be seen from $(5.3)_{2,3}$ that r_1 and r_2 are real if and only if $\lambda \geq 2$, so that solutions of the form (5.1) are valid only for $\lambda > 2$, which is assured by (3.21).

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Абстракт — Тонкая, цилиндрическая, упругая труба, усиленная на внешней поверхности двух параметрических семейством нерастяжимых канатов, деформируется таким образом, что внутренная поверхность принимает заданную форму. Предыдущие решение этой задачи, важное только, когда кривизна деформированной внутренной поверхности непрерывна всегда, обобщено к более общему случаю, для которого кривизна обладает разрывами. Применяется метод подобранных асимптомических разложений с целью построения пещения в окружности разрывов и с целью связания его гладко с раньшем решением, которое, как, указано, сохраняет важность далеко от разрывов.

Описанная здесь деформация происходит на пример, когда усиленная упругая труба деформирована вследствие заключенной жестко прессформы или оправки и находится под влиянием внешнего давления.